ORBIFOLDS OF LATTICE VERTEX ALGEBRAS UNDER AN ISOMETRY OF ORDER TWO

BOJKO BAKALOV AND JASON ELSINGER

Abstract. Every isometry σ of a positive-definite even lattice Q can be lifted to an automorphism of the lattice vertex algebra V_Q. An important problem in vertex algebra theory and conformal field theory is to classify the representations of the σ-invariant subalgebra V_Q^σ of V_Q, known as an orbifold. In the case when σ is an isometry of Q of order two, we classify the irreducible modules of the orbifold vertex algebra V_Q^σ and identify them as submodules of twisted or untwisted V_Q-modules. The examples where Q is a root lattice and σ is a Dynkin diagram automorphism are presented in detail.

1. Introduction

The notion of a vertex algebra introduced by Borcherds [B1] provides a rigorous algebraic description of two-dimensional chiral conformal field theory (see e.g. [BPZ, Go, DMS]), and is a powerful tool for studying representations of infinite-dimensional Lie algebras. The theory of vertex algebras has been developed in [FLM2, K2, FB, LL, KRR] among other works. One of the spectacular early applications of vertex algebras was Borcherds' proof of the moonshine conjectures about the Monster group (see [B2, Ga]), which used essentially the Frenkel–Lepowsky–Meurman construction of a vertex algebra with a natural action of the Monster on it [FLM2].

The FLM vertex algebra was constructed in three steps (see [FLM2, Ga]). First, one constructs a vertex algebra V_Q from any even lattice Q (see [B1, FLM2]) by generalizing the Frenkel–Kac realization of affine Kac–Moody algebras in terms of vertex operators [FK]. Second, given an isometry σ of Q, one lifts it to an automorphism of V_Q and constructs the so-called σ-twisted V_Q-modules [D2, FFR] that axiomatize the properties of twisted vertex operators [KP, Le, FLM1]. Then the FLM vertex algebra is defined as the direct sum of the σ-invariant

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part V_Q^σ and one of the σ-twisted V_Q-modules, in the special case when $\sigma = -1$ and Q is the Leech lattice.

In general, if σ is an automorphism of a vertex algebra V, the σ-invariants V^σ form a subalgebra of V known as an orbifold, which is important in conformal field theory (see e.g. [DVVV, KT, DLM1] among many other works). Every σ-twisted representation of V becomes untwisted when restricted to V^σ. It is a long-standing conjecture that all irreducible V^σ-modules are obtained by restriction from twisted or untwisted V-modules. Under certain assumptions, this conjecture has been proved recently by M. Miyamoto [M1, M2, M3], and he has also shown that the vertex algebra V^σ is rational, i.e., every (admissible) module is a direct sum of irreducible ones.

In this paper, we are concerned with the case when Q is a positive-definite even lattice and σ is an isometry of Q of order two. We classify and construct explicitly all irreducible modules of the orbifold vertex algebra V_Q^σ, and we realize them as submodules of twisted or untwisted V_Q-modules. In the important special case when $\sigma = -1$, the classification was done previously by Dong–Nagatomo and Abe–Dong [DN, AD]. Our approach is to restrict from V_Q^σ to V_L^σ where L is the sublattice of Q spanned by eigenvectors of σ. The subalgebra V_L^σ factors as a tensor product $V_{L_+} \otimes V_{L_-}^\sigma$, where $L_\pm$ consists of $\alpha \in Q$ with $\sigma\alpha = \pm\alpha$, respectively. By the results of [DLM2, ABD, DJL], every V_L^σ-module is a direct sum of irreducible ones. We describe explicitly the irreducible V_L^σ-modules using [FHL, DN, AD], and then utilize the intertwining operators among them [A1, ADL] to determine the irreducible V_Q^σ-modules. A new ingredient is the introduction of a sublattice $\bar{Q}$ of Q such that $V_Q^\sigma = V_{\bar{Q}}^\sigma$ and σ has order 2 as an automorphism of $V_{\bar{Q}}$ (while in general σ has order 4 on V_Q).

Our work was motivated by the examples when Q is the root lattice of a simply-laced simple Lie algebra and σ is a Dynkin diagram automorphism, which are related to twisted affine Kac–Moody algebras [K1]. We also consider the case when σ is the negative of a Dynkin diagram automorphism, which is relevant to the Slodowy generalized intersection matrix Lie algebras [S, Be].

The paper is organized as follows. In Section 2, we briefly review lattice vertex algebras and their twisted modules, and the results for $\sigma = -1$ that we need. Our main results are presented in Section 3. The examples when Q is a root lattice of type ADE are discussed in detail in Section 4.

2. Vertex algebras and their twisted modules

In this section, we briefly review lattice vertex algebras and their twisted modules. We also recall the case when $\sigma = -1$, which will be used essentially for the general case. Good general references on vertex algebras are [FLM2, K2, FB, LL, KRR].

2.1. **Vertex algebras.** Recall that a *vertex algebra* is a vector space V with a distinguished vector $\mathbf{1} \in V$ (vacuum vector), together with a linear map (state-field correspondence)

$$Y(\cdot, z)\cdot \colon V \otimes V \to V(\!(z)\!) = V[[z]][z^{-1}] \, . \tag{2.1}$$

Thus, for every $a \in V$, we have the *field* $Y(a, z)\colon V \to V(\!(z)\!)$. This field can be viewed as a formal power series from $(\operatorname{End} V)[[z, z^{-1}]]$, which involves only finitely many negative powers of z when applied to any vector.

The coefficients in front of powers of z in this expansion are known as the *modes* of a:

$$Y(a, z) = \sum_{n\in\mathbb{Z}} a_{(n)} \, z^{-n-1} \, , \qquad a_{(n)} \in \operatorname{End} V \, . \tag{2.2}$$

The vacuum vector $\mathbf{1}$ plays the role of an identity in the sense that

$$a_{(-1)}\mathbf{1} = \mathbf{1}_{(-1)}a = a \, , \qquad a_{(n)}\mathbf{1} = 0 \, , \quad n \geq 0 \, .$$

In particular, $Y(a, z)\mathbf{1} \in V[[z]]$ is regular at $z = 0$, and its value at $z = 0$ is equal to a.

The main axiom for a vertex algebra is the *Borcherds identity* (also called Jacobi identity [FLM2]) satisfied by the modes:

$$\begin{aligned} \sum_{j=0}^{\infty} \binom{m}{j} (a_{(n+j)}b)_{(k+m-j)}c &= \sum_{j=0}^{\infty} \binom{n}{j} (-1)^j a_{(m+n-j)}(b_{(k+j)}c) \\ &- \sum_{j=0}^{\infty} \binom{n}{j} (-1)^{j+n} \, b_{(k+n-j)}(a_{(m+j)}c) \, , \end{aligned} \tag{2.3}$$

where $a, b, c \in V$. Note that the above sums are finite, because $a_{(n)}b = 0$ for sufficiently large n.

We say that a vertex algebra V is (strongly) *generated* by a subset $S \subset V$ if V is linearly spanned by the vacuum $\mathbf{1}$ and all elements of the form $a_{1(n_1)} \cdots a_{k(n_k)}\mathbf{1}$, where $k \geq 1$, $a_i \in S$, $n_i < 0$.

2.2. **Twisted representations of vertex algebras.** A *representation* of a vertex algebra V, or a *V-module*, is a vector space M endowed with a linear map $Y(\cdot,z)\cdot\colon V\otimes M\to M(\!(z)\!)$ (cf. (2.1), (2.2)) such that the Borcherds identity (2.3) holds for $a,b\in V$, $c\in M$ (see [FB, LL, KRR]).

Now let σ be an automorphism of V of finite order r. Then σ is diagonalizable. In the definition of a *σ-twisted representation* M of V [FFR, D2], the image of the above map Y is allowed to have nonintegral rational powers of z, so that

$$Y(a,z)=\sum_{n\in p+\mathbb{Z}} a_{(n)}\, z^{-n-1}\,, \qquad \text{if} \quad \sigma a = e^{-2\pi \mathrm{i} p} a\,,\ p\in\frac{1}{r}\mathbb{Z}\,,$$

where $a_{(n)}\in \operatorname{End} M$. The Borcherds identity (2.3) satisfied by the modes remains the same in the twisted case, provided that a is an eigenvector of σ. An important consequence of the Borcherds identity is the *locality* property [DL, Li]:

$$(z-w)^N[Y(a,z),Y(b,w)]=0 \tag{2.4}$$

for sufficiently large N depending on a,b (one can take N to be such that $a_{(n)}b=0$ for $n\geq N$).

The following result provides a rigorous interpretation of the *operator product expansion* in conformal field theory (cf. [Go, DMS]) in the case of twisted modules.

Proposition 2.1 ([BM])**.** *Let V be a vertex algebra, σ an automorphism of V, and M a σ-twisted representation of V. Then*

$$\frac{1}{k!}\partial_z^k\Bigl((z-w)^N\,Y(a,z)Y(b,w)c\Bigr)\Big|_{z=w} = Y(a_{(N-1-k)}b,w)c \tag{2.5}$$

for all $a,b\in V$, $c\in M$, $k\geq 0$, and sufficiently large N. Conversely, (2.4) *and* (2.5) *imply the Borcherds identity* (2.3)*.*

Recall from [FHL] that if V_1 and V_2 are vertex algebras, their tensor product is again a vertex algebra with

$$Y(v_1\otimes v_2,z)=Y(v_1,z)\otimes Y(v_2,z)\,, \qquad v_i\in V_i\,.$$

Furthermore, if M_i is a V_i-module, then the above formula defines the structure of a $(V_1\otimes V_2)$-module on $M_1\otimes M_2$ (see [FHL]). This is also true for twisted modules.

Lemma 2.2. *For $i=1,2$, let V_i be a vertex algebra, σ_i an automorphism of V_i, and M_i a σ_i-twisted representation of V_i. Then $M_1\otimes M_2$ is a $(\sigma_1\otimes\sigma_2)$-twisted module over $V_1\otimes V_2$.*

Proof. By Proposition 2.1, it is enough to check (2.4) and (2.5) for $a = a_1 \otimes a_2$ and $b = b_1 \otimes b_2$, given that they hold for the pairs $a_1, b_1 \in V_1$ and $a_2, b_2 \in V_2$. This is done by a straightforward calculation. □

2.3. **Lattice vertex algebras.** Let Q be an *integral lattice*, i.e., a free abelian group of finite rank equipped with a symmetric nondegenerate bilinear form $(\cdot|\cdot)\colon Q \times Q \to \mathbb{Z}$. We will assume that Q is *even*, i.e., $|\alpha|^2 = (\alpha|\alpha) \in 2\mathbb{Z}$ for all $\alpha \in Q$. We denote by $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} Q$ the corresponding complex vector space considered as an abelian Lie algebra, and extend the bilinear form to it.

The *Heisenberg algebra* $\hat{\mathfrak{h}} = \mathfrak{h}[t, t^{-1}] \oplus \mathbb{C}K$ is the Lie algebra with brackets

$$[a_m, b_n] = m\delta_{m,-n}(a|b)K\,, \qquad a_m = at^m\,, \tag{2.6}$$

where K is central. Its irreducible highest-weight representation

$$\mathcal{F} = \operatorname{Ind}^{\hat{\mathfrak{h}}}_{\mathfrak{h}[t]\oplus\mathbb{C}K} \mathbb{C} \cong S(\mathfrak{h}[t^{-1}]t^{-1})$$

on which $K = 1$ is known as the (bosonic) *Fock space*.

Following [FK, B1], we consider a 2-cocycle $\varepsilon\colon Q \times Q \to \{\pm 1\}$ such that

$$\varepsilon(\alpha, \alpha) = (-1)^{|\alpha|^2(|\alpha|^2+1)/2}\,, \qquad \alpha \in Q\,, \tag{2.7}$$

and the associative algebra $\mathbb{C}_\varepsilon[Q]$ with basis $\{e^\alpha\}_{\alpha\in Q}$ and multiplication

$$e^\alpha e^\beta = \varepsilon(\alpha, \beta)e^{\alpha+\beta}\,. \tag{2.8}$$

Such a 2-cocycle ε is unique up to equivalence and can be chosen to be bimultiplicative. In addition,

$$\varepsilon(\alpha, \beta)\varepsilon(\beta, \alpha) = (-1)^{(\alpha|\beta)}\,, \qquad \alpha, \beta \in Q\,. \tag{2.9}$$

The *lattice vertex algebra* associated to Q is defined as $V_Q = \mathcal{F} \otimes \mathbb{C}_\varepsilon[Q]$, where the vacuum vector is $\mathbf{1} = 1 \otimes e^0$. We let the Heisenberg algebra act on V_Q so that

$$a_n e^\beta = \delta_{n,0}(a|\beta)e^\beta\,, \quad n \geq 0\,, \qquad a \in \mathfrak{h}\,.$$

The state-field correspondence on V_Q is uniquely determined by the generating fields:

$$Y(a_{-1}\mathbf{1}, z) = \sum_{n\in\mathbb{Z}} a_n\, z^{-n-1}\,, \qquad a \in \mathfrak{h}\,, \tag{2.10}$$

$$Y(e^\alpha, z) = e^\alpha z^{\alpha_0} \exp\Big(\sum_{n<0} \alpha_n \frac{z^{-n}}{-n}\Big) \exp\Big(\sum_{n>0} \alpha_n \frac{z^{-n}}{-n}\Big)\,, \tag{2.11}$$

where $z^{\alpha_0} e^\beta = z^{(\alpha|\beta)} e^\beta$.

Notice that $\mathcal{F} \subset V_Q$ is a vertex subalgebra, which we call the *Heisenberg vertex algebra*. The map $\mathfrak{h} \to \mathcal{F}$ given by $a \mapsto a_{-1}\mathbf{1}$ is injective. From now on, we will slightly abuse the notation and identify $a \in \mathfrak{h}$ with $a_{-1}\mathbf{1} \in \mathcal{F}$; then $a_{(n)} = a_n$ for all $n \in \mathbb{Z}$.

2.4. **Twisted Heisenberg algebra.** Every automorphism σ of $\mathfrak{h}$ preserving the bilinear form induces automorphisms of $\hat{\mathfrak{h}}$ and $\mathcal{F}$, which will be denoted again as σ. As before, assume that σ has a finite order r. The action of σ can be extended to $\mathfrak{h}[t^{1/r}, t^{-1/r}] \oplus \mathbb{C}K$ by letting

$$\sigma(at^m) = \sigma(a) e^{2\pi \mathrm{i} m} t^m \,, \quad \sigma(K) = K \,, \qquad a \in \mathfrak{h} \,, \; m \in \frac{1}{r}\mathbb{Z} \,.$$

The *σ-twisted Heisenberg algebra* $\hat{\mathfrak{h}}_\sigma$ is defined as the set of all σ-invariant elements (see e.g. [KP, Le, FLM1]). In other words, $\hat{\mathfrak{h}}_\sigma$ is spanned over $\mathbb{C}$ by K and the elements $a_m = at^m$ such that $\sigma a = e^{-2\pi \mathrm{i} m} a$. This is a Lie algebra with bracket (cf. (2.6))

$$[a_m, b_n] = m \delta_{m,-n} (a|b) K \,, \qquad a, b \in \mathfrak{h} \,, \;\; m, n \in \frac{1}{r}\mathbb{Z} \,.$$

Let $\hat{\mathfrak{h}}_\sigma^{\geq}$ (respectively, $\hat{\mathfrak{h}}_\sigma^{<}$) be the abelian subalgebra of $\hat{\mathfrak{h}}_\sigma$ spanned by all elements a_m with $m \geq 0$ (respectively, $m < 0$).

The *σ-twisted Fock space* is defined as

$$\mathcal{F}_\sigma = \operatorname{Ind}_{\hat{\mathfrak{h}}_\sigma^{\geq} \oplus \mathbb{C}K}^{\hat{\mathfrak{h}}_\sigma} \mathbb{C} \cong S(\hat{\mathfrak{h}}_\sigma^{<}) \,, \tag{2.12}$$

where $\hat{\mathfrak{h}}_\sigma^{\geq}$ acts on $\mathbb{C}$ trivially and K acts as the identity operator. Then $\mathcal{F}_\sigma$ is an irreducible highest-weight representation of $\hat{\mathfrak{h}}_\sigma$, and has the structure of a σ-twisted representation of the vertex algebra $\mathcal{F}$ (see e.g. [FLM2, KRR]). This structure can be described as follows. We let $Y(\mathbf{1}, z)$ be the identity operator and

$$Y(a, z) = \sum_{n \in p + \mathbb{Z}} a_n \, z^{-n-1} \,, \qquad a \in \mathfrak{h} \,, \;\; \sigma a = e^{-2\pi \mathrm{i} p} a \,, \;\; p \in \frac{1}{r}\mathbb{Z} \,,$$

and we extend Y to all $a \in \mathfrak{h}$ by linearity. The action of Y on other elements of $\mathcal{F}$ is then determined by applying several times the product formula (2.5). More explicitly, $\mathcal{F}$ is spanned by elements of the form $a^1_{m_1} \cdots a^k_{m_k} \mathbf{1}$ where $a^j \in \mathfrak{h}$, and we have:

$$\begin{aligned} & Y(a^1_{m_1} \cdots a^k_{m_k} \mathbf{1}, z) c \\ & \quad = \prod_{j=1}^k \partial_{z_j}^{(N-1-m_j)} \Bigl(\prod_{j=1}^k (z_j - z)^N \, Y(a^1, z_1) \cdots Y(a^k, z_k) c \Bigr) \Big|_{z_1 = \cdots = z_k = z} \end{aligned}$$

for all $c \in \mathcal{F}_\sigma$ and sufficiently large N. In the above formula, we use the divided-power notation $\partial^{(n)} = \partial^n/n!$.

2.5. **Twisted representations of lattice vertex algebras.** Let σ be an automorphism (or *isometry*) of the lattice Q of finite order r, so that

$$(\sigma\alpha|\sigma\beta) = (\alpha|\beta)\,, \qquad \alpha,\beta \in Q\,. \tag{2.13}$$

The uniqueness of the cocycle ε and (2.13), (2.9) imply that

$$\eta(\alpha+\beta)\varepsilon(\sigma\alpha,\sigma\beta) = \eta(\alpha)\eta(\beta)\varepsilon(\alpha,\beta) \tag{2.14}$$

for some function $\eta\colon Q \to \{\pm 1\}$.

Lemma 2.3. *Let L be a sublattice of Q such that $\varepsilon(\sigma\alpha,\sigma\beta) = \varepsilon(\alpha,\beta)$ for $\alpha,\beta \in L$. Then there exists a function $\eta\colon Q \to \{\pm 1\}$ satisfying (2.14) and $\eta(\alpha) = 1$ for all $\alpha \in L$.*

Proof. First observe that, by (2.7) and (2.13), (2.14) for $\alpha = \beta$, we have $\eta(2\alpha) = 1$ for all $\alpha \in Q$. Since, by bimultiplicativity, $\varepsilon(2\alpha,\beta) = 1$, we obtain that $\eta(2\alpha+\beta) = \eta(\beta)$ for all α,β. Therefore, η is defined on $Q/2Q$. If $\alpha_1,\dots,\alpha_\ell$ is any $\mathbb{Z}$-basis for Q, we can set all $\eta(\alpha_i) = 1$ and then η is uniquely extended to the whole Q by (2.14). We can pick a $\mathbb{Z}$-basis for Q so that $d_1\alpha_1,\dots,d_m\alpha_m$ is a $\mathbb{Z}$-basis for L, where $m \le \ell$ and $d_i \in \mathbb{Z}$. Then the extension of η to Q will satisfy $\eta(\alpha) = 1$ for all $\alpha \in L$. ∎

In particular, η can be chosen such that

$$\eta(\alpha) = 1\,, \qquad \alpha \in Q \cap \mathfrak{h}_0\,, \tag{2.15}$$

where $\mathfrak{h}_0$ denotes the subspace of $\mathfrak{h}$ consisting of vectors fixed under σ. Then σ can be lifted to an automorphism of the lattice vertex algebra V_Q by setting

$$\sigma(a_n) = \sigma(a)_n\,, \quad \sigma(e^\alpha) = \eta(\alpha)e^{\sigma\alpha}\,, \qquad a \in \mathfrak{h}\,,\ \alpha \in Q\,. \tag{2.16}$$

Remark 2.4. The order of σ is r or $2r$ when viewed as an automorphism of V_Q, where r is the order of σ on Q. The set $\bar{Q}$ of $\alpha \in Q$ such that $\sigma^r(e^\alpha) = e^\alpha$ is a sublattice of Q of index 1 or 2, and we have $(V_Q)^{\sigma^r} = V_{\bar{Q}}$.

We will now recall the construction of irreducible σ-twisted V_Q-modules (see [KP, Le, D2, BK]). Introduce the group $G = \mathbb{C}^\times \times \exp\mathfrak{h}_0 \times Q$ consisting of elements $c\,e^h U_\alpha$ ($c \in \mathbb{C}^\times$, $h \in \mathfrak{h}_0$, $\alpha \in Q$) with

multiplication

$$\begin{aligned} e^h e^{h'} &= e^{h+h'} , \\ e^h U_\alpha e^{-h} &= e^{(h|\alpha)} U_\alpha , \\ U_\alpha U_\beta &= \varepsilon(\alpha,\beta) B_{\alpha,\beta}^{-1} U_{\alpha+\beta} , \end{aligned}$$

where

$$B_{\alpha,\beta} = r^{-(\alpha|\beta)} \prod_{k=1}^{r-1} \bigl(1 - e^{2\pi \mathrm{i} k/r}\bigr)^{(\sigma^k \alpha|\beta)} .$$

Let

$$C_\alpha = \eta(\alpha) U_{\sigma\alpha}^{-1} U_\alpha e^{2\pi\mathrm{i}(b_\alpha + \pi_0\alpha)} , \qquad b_\alpha = \frac{1}{2}\bigl(|\pi_0\alpha|^2 - |\alpha|^2\bigr) ,$$

where π_0 is the projection of $\mathfrak{h}$ onto $\mathfrak{h}_0$. Then $C_\alpha C_\beta = C_{\alpha+\beta}$ and all C_α are in the center of G. We define G_σ to be the quotient group $G/\{C_\alpha\}_{\alpha\in Q}$.

Consider an irreducible representation Ω of G_σ. Such representations are parameterized by the set $(Q^*/Q)^\sigma$ of σ-invariants in Q^*/Q, i.e., by $\lambda + Q$ such that $\lambda \in Q^*$ and $(1-\sigma)\lambda \in Q$ (see [BK, Proposition 4.4]). Furthermore, the action of $\exp \mathfrak{h}_0$ on Ω is semisimple:

$$\Omega = \bigoplus_{\mu\in\pi_0(Q^*)} \Omega_\mu ,$$

where

$$\Omega_\mu = \{ v \in \Omega \,|\, e^h v = e^{(h|\mu)} v \ \text{ for } \ h \in \mathfrak{h}_0 \} .$$

Then $\mathcal{F}_\sigma \otimes \Omega$ is an irreducible σ-twisted V_Q-module with an action defined as follows. We define $Y(a,z)$ for $a \in \mathfrak{h}$ as before, and for $\alpha \in Q$ we let

(2.17)

$$Y(e^\alpha, z) = \exp\Biggl(\sum_{n\in\frac{1}{r}\mathbb{Z}_{<0}} \alpha_n \frac{z^{-n}}{-n} \Biggr) \exp\Biggl(\sum_{n\in\frac{1}{r}\mathbb{Z}_{>0}} \alpha_n \frac{z^{-n}}{-n} \Biggr) \otimes U_\alpha z^{b_\alpha + \pi_0\alpha} .$$

Here the action of $z^{\pi_0\alpha}$ is given by $z^{\pi_0\alpha} v = z^{(\pi_0\alpha|\mu)} v$ for $v \in \Omega_\mu$, and $(\pi_0\alpha|\mu) \in \frac{1}{r}\mathbb{Z}$. The action of Y on all of V_Q can be obtained by applying the product formula (2.5).

By [BK, Theorem 4.2], every irreducible σ-twisted V_Q-module is obtained in this way, and every σ-twisted V_Q-module is a direct sum of irreducible ones. In the special case when $\sigma = 1$, we get Dong's Theorem that the irreducible V_Q-modules are classified by Q^*/Q (see [D1]). Explicitly, they are given by:

$$V_{\lambda+Q} = \mathcal{F} \otimes \mathbb{C}_\varepsilon[Q] e^\lambda , \qquad \lambda \in Q^* .$$

When the lattice Q is written as an orthogonal direct sum of sublattices, $Q = L_1 \oplus L_2$, we have a natural isomorphism $V_Q \cong V_{L_1} \otimes V_{L_2}$. The following lemma shows that if L_1 and L_2 are σ-invariant, there is a correspondence of irreducible twisted modules (cf. [FHL] and Lemma 2.2).

Lemma 2.5. *Let Q be an even lattice, $Q = L_1 \oplus L_2$ an orthogonal direct sum, and σ an automorphism of Q such that $\sigma(L_i) \subseteq L_i$ $(i = 1, 2)$. Set $\sigma_i = \sigma|_{L_i}$. Then every irreducible σ-twisted V_Q-module M is a tensor product, $M \cong M_1 \otimes M_2$, where M_i is an irreducible σ_i-twisted V_{L_i}-module.*

Proof. This follows from the classification of irreducible σ-twisted V_Q-modules described above. Indeed, $Q = L_1 \oplus L_2$ gives rise to a decomposition of the Heisenberg Lie algebra as a direct sum $\hat{\mathfrak{h}} = \hat{\mathfrak{h}}_1 \oplus \hat{\mathfrak{h}}_2$. We also have a similar decomposition for the corresponding twisted Heisenberg algebras. Then for the twisted Fock spaces, we get $\mathcal{F}_\sigma = \mathcal{F}_{\sigma_1} \otimes \mathcal{F}_{\sigma_2}$. Similarly, the group G is a direct product of its subgroups G_1 and G_2 associated to the lattices L_1 and L_2, respectively. □

2.6. **The case $\sigma = -1$.** Now we will review what is known in the case when $\sigma = -1$, which will be used essentially in our treatment of the general case. In this subsection, we will denote the even integral lattice by L instead of Q. As before, let $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} L$ be the corresponding complex vector space.

Observe that $\mathfrak{h}_0 = 0$, $\pi_0 = 0$, and we can assume that $\eta(\alpha) = 1$ for all $\alpha \in L$. Hence, the automorphism σ acts on V_L by

$$\sigma\left(h^1_{(-n_1)} \cdots h^k_{(-n_k)} e^\alpha\right) = (-1)^k h^1_{(-n_1)} \cdots h^k_{(-n_k)} e^{-\alpha} \tag{2.18}$$

for $h^i \in \mathfrak{h}$, $n_i \in \mathbb{Z}_{\geq 0}$ and $\alpha \in L$. The group G consists of cU_α ($c \in \mathbb{C}^\times, \alpha \in L$), and its center consists of cU_α with $\alpha \in 2L^* \cap L$. Then G_σ is the quotient of G by $\{U_\alpha^{-1} U_{-\alpha}\}_{\alpha \in L}$. The twisted Heisenberg algebra is $\hat{\mathfrak{h}}_\sigma = \mathfrak{h}[t, t^{-1}] t^{1/2} \oplus \mathbb{C}K$.

For any G_σ-module T, define $V_L^T = \mathcal{F}_\sigma \otimes T$, where $\mathcal{F}_\sigma$ is the twisted Fock space (cf. (2.12)). By [D2], every irreducible σ-twisted V_L-module is isomorphic to V_L^T for some irreducible G_σ-module T on which cU_0 acts as cI, where I is the identity operator. Such modules T can be described equivalently as G-modules, on which $cU_0 = cI$ and $U_\alpha = U_{-\alpha}$. The irreducible ones are determined by the central characters χ such that $\chi(U_\alpha) = \chi(U_{-\alpha})$ for $\alpha \in 2L^* \cap L$. We have:

$$U_\alpha^2 = U_\alpha U_{-\alpha} = \varepsilon(\alpha, -\alpha) B_{\alpha,-\alpha}^{-1} U_0 \,,$$

which implies that

$$\chi(U_\alpha) = s(\alpha) e^{\pi \mathrm{i} |\alpha|^2 (|\alpha|^2+1)/2} 2^{-|\alpha|^2}\,, \tag{2.19}$$

where $s(\alpha) \in \{\pm 1\}$ satisfies $s(\alpha+\beta) = s(\alpha)s(\beta)$. All such maps s have the form

$$s(\alpha) = (-1)^{(\alpha|\mu)}$$

for some $\mu \in (2L^* \cap L)^*$. The corresponding central characters χ will be denoted as χ_μ, and the corresponding G_σ-module T as T_μ.

We define an action of σ on V_L^T by

$$\sigma\left(h^1_{(-n_1)} \cdots h^k_{(-n_k)} t\right) = (-1)^k h^1_{(-n_1)} \cdots h^k_{(-n_k)} t \tag{2.20}$$

for $h^i \in \mathfrak{h}$, $n_i \in \frac{1}{2} + \mathbb{Z}_{\geq 0}$ and $t \in T$. The eigenspaces for σ are denoted $V_L^{T,\pm}$. Then we have

$$\sigma\bigl(Y(a,z)v\bigr) = Y(\sigma a, z)(\sigma v)\,, \qquad a \in V_L\,,\ v \in V_L^T\,, \tag{2.21}$$

which implies that σ is an automorphism of V_L^σ-modules. In particular, $V_L^{T,\pm}$ are V_L^σ-modules. Similarly, we define an action of σ on the untwisted V_L-modules $V_{\lambda+L}$ by (2.18) for $\alpha \in \lambda + L$. Note that $\sigma V_{\lambda+L} \subseteq V_{-\lambda+L}$. Hence, if $\lambda \in L^*$, $2\lambda \in L$, then the eigenspaces $V^\pm_{\lambda+L}$ are V_L^σ-modules. On the other hand, if $2\lambda \notin L$, then $V_{\lambda+L} \cong V_{-\lambda+L}$ as V_L^σ-modules.

Theorem 2.6 ([DN, AD])**.** *Let L be a positive-definite even lattice and $\sigma = -1$ on L. Then any irreducible admissible V_L^σ-module is isomorphic to one of the following*:

$$V^\pm_{\lambda+L} \quad (\lambda \in L^*,\, 2\lambda \in L), \quad V_{\lambda+L} \quad (\lambda \in L^*,\, 2\lambda \notin L), \quad V_L^{T,\pm},$$

where T is an irreducible G_σ-module.

Next, we will discuss intertwining operators between the irreducible V_L^σ-modules. For a vector space U, denote by

$$U\{z\} = \Bigl\{ \sum_{n\in\mathbb{Q}} v_{(n)} z^{-n-1} \Bigm| v_{(n)} \in U \Bigr\}$$

the space of U-valued formal series involving rational powers of z. Let V be a vertex algebra, and M_1, M_2, M_3 be V-modules, which are not necessarily distinct. Recall from [FHL] that an *intertwining operator* of type $\begin{pmatrix} M_3 \\ M_1\ M_2 \end{pmatrix}$ is a linear map $\mathcal{Y}\colon M_1 \otimes M_2 \to M_3\{z\}$, or equivalently,

$$\mathcal{Y}\colon M_1 \to \mathrm{Hom}(M_2, M_3)\{z\}\,,$$
$$v \mapsto \mathcal{Y}(v,z) = \sum_{n\in\mathbb{Q}} v_{(n)} z^{-n-1}\,, \quad v_{(n)} \in \mathrm{Hom}(M_2, M_3)$$

such that $v_{(n)}u = 0$ for $n \gg 0$, and the Borcherds identity (2.3) holds for $a \in V, b \in M_1$ and $c \in M_2$ with $k \in \mathbb{Q}$ and $m, n \in \mathbb{Z}$. The intertwining operators of type $\binom{M_3}{M_1\ M_2}$ form a vector space denoted $\mathcal{V}^{M_3}_{M_1\,M_2}$. The *fusion rule* associated with an algebra V and its modules M_1, M_2, M_3 is $N^{M_3}_{M_1\,M_2} = \dim \mathcal{V}^{M_3}_{M_1\,M_2}$.

The fusion rules for V_L^σ and its irreducible modules were calculated in [A1, ADL] to be either zero or one. In order to present their theorem, we first introduce some additional notation. We assume that $\lambda \in L^*$ is such that $2\lambda \in L$, and we let

$$\pi_{\lambda,\mu} = (-1)^{|\lambda|^2|\mu|^2}\,, \qquad \lambda, \mu \in L^*\,, \tag{2.22}$$

$$c_\chi(\lambda) = (-1)^{(\lambda|2\lambda)}\varepsilon(\lambda, 2\lambda)s(2\lambda)\,. \tag{2.23}$$

For any central character χ of G_σ, let $\chi^{(\lambda)}$ be the central character defined by

$$\chi^{(\lambda)}(U_\alpha) = (-1)^{(\alpha|\lambda)}\chi(U_\alpha)\,, \tag{2.24}$$

and set $T_\chi^{(\lambda)} = T_{\chi^{(\lambda)}}$. Note that when $\chi = \chi_\mu$ and $T = T_\mu$, we have $\chi_\mu^{(\lambda)} = \chi_{\lambda+\mu}$ and $T_\mu^{(\lambda)} = T_{\lambda+\mu}$.

The following theorem is a special case of Theorem 5.1 from [ADL].

Theorem 2.7 ([ADL]). *Let L be a positive-definite even lattice and $\lambda \in L^* \cap \frac{1}{2}L$. Then for two irreducible V_L^σ-modules M_2, M_3 and for $\epsilon \in \{\pm\}$, the fusion rule of type $\binom{M_3}{V^\epsilon_{\lambda+L}\ M_2}$ is equal to 1 if and only if the pair (M_2, M_3) is one of the following*:

$$(V_{\mu+L}, V_{\lambda+\mu+L}), \quad \mu \in L^*,\ \ 2\mu \notin L,$$
$$(V^{\epsilon_1}_{\mu+L}, V^{\epsilon_2}_{\lambda+\mu+L}), \quad \mu \in L^*,\ \ 2\mu \in L,\ \ \epsilon_1 \in \{\pm\},\ \ \epsilon_2 = \epsilon_1\epsilon\pi_{\lambda,2\mu},$$
$$(V_L^{T_\chi,\epsilon_1}, V_L^{T_\chi^{(\lambda)},\epsilon_2}), \quad \epsilon_1 \in \{\pm\},\ \ \epsilon_2 = c_\chi(\lambda)\epsilon_1\epsilon.$$

In all other cases, the fusion rules of type $\binom{M_3}{V^\epsilon_{\lambda+L}\ M_2}$ are zero.

3. Classification of irreducible modules

In this section, we prove our main result, the classification of all irreducible modules over the orbifold vertex algebra V_Q^σ. As before, Q is a positive-definite even integral lattice and σ is an isometry of Q of order 2.

3.1. **The sublattice $\bar{Q}$.** The map σ is extended by linearity to the complex vector space $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} Q$. We will denote by

$$\pi_\pm = \frac{1}{2}(1 \pm \sigma)\,, \qquad \alpha_\pm = \pi_\pm(\alpha) \tag{3.1}$$

the projections onto the eigenspaces of σ. Introduce the important sublattices

$$L_\pm = \mathfrak{h}_\pm \cap Q, \qquad L = L_+ \oplus L_- \subseteq Q\,, \tag{3.2}$$

where $\mathfrak{h}_\pm = \pi_\pm(\mathfrak{h})$. Note that $\mathfrak{h} = \mathfrak{h}_+ \oplus \mathfrak{h}_-$ is an orthogonal direct sum.

Lemma 3.1. *We have $\alpha_\pm \in (L_\pm)^* \subseteq L^*$ for any $\alpha \in Q$.*

Proof. Indeed,

$$(\alpha_+|\beta) = (\alpha_+ + \alpha_-|\beta) = (\alpha|\beta) \in \mathbb{Z}$$

for all $\alpha \in Q$, $\beta \in L_+ \subseteq Q$. □

Observe that $2\alpha_\pm \in L_\pm$ and $|\alpha_\pm|^2 = \frac{1}{4}|2\alpha_\pm|^2 \in \frac{1}{2}\mathbb{Z}$ for all $\alpha \in Q$.

Lemma 3.2. *For $\alpha \in Q$, the following are equivalent*:

(*i*) $\sigma^2(e^\alpha) = e^\alpha$,
(*ii*) $|\alpha_\pm|^2 \in \mathbb{Z}$,
(*iii*) $(\alpha|\sigma\alpha) \in 2\mathbb{Z}$.

Proof. Note that $4|\alpha_\pm|^2 = |\alpha \pm \sigma\alpha|^2 = 2|\alpha|^2 \pm 2(\alpha|\sigma\alpha)$, so that

$$|\alpha_\pm|^2 = \frac{1}{2}(\alpha|\sigma\alpha) \mod \mathbb{Z}.$$

This shows the equivalence between (*ii*) and (*iii*).

Using (2.16), we find $\sigma^2(e^\alpha) = \eta(\alpha)\eta(\sigma\alpha)e^\alpha$. On the other hand, by (2.9), (2.14) and (2.15), we have

$$\eta(\alpha)\eta(\sigma\alpha) = \varepsilon(\alpha, \sigma\alpha)\varepsilon(\sigma\alpha, \alpha) = (-1)^{(\alpha|\sigma\alpha)}.$$

This proves the equivalence between (*i*) and (*iii*). □

From now on, we let

$$\bar{Q} = \{\alpha \in Q \,|\, (\alpha|\sigma\alpha) \in 2\mathbb{Z}\}\,. \tag{3.3}$$

It is clear that $\bar{Q}$ is σ-invariant.

Lemma 3.3. *The subset $\bar{Q}$ is a sublattice of Q of index* 1 *or* 2*.*

Proof. For $\alpha, \beta \in Q$, we have

$$(\alpha - \beta|\sigma\alpha - \sigma\beta) = (\alpha|\sigma\alpha) + (\beta|\sigma\beta) \mod 2\mathbb{Z}\,,$$

since

$$(\alpha|\sigma\beta) = (\sigma\alpha|\sigma^2\beta) = (\beta|\sigma\alpha)\,.$$

Now if $\alpha, \beta \in \bar{Q}$ or $\alpha, \beta \notin \bar{Q}$, then $\alpha - \beta \in \bar{Q}$. □

As a consequence of Lemmas 3.1 and 3.2, we obtain:

Corollary 3.4. *The lattices $\mathbb{Z}\alpha_\pm + L$ are integral for all $\alpha \in \bar{Q}$.*

By definition, we have $(V_Q)^{\sigma^2} = V_{\bar{Q}}$, and

$$V_Q^\sigma = \big((V_Q)^{\sigma^2}\big)^\sigma = V_{\bar{Q}}^\sigma \,. \tag{3.4}$$

Therefore, we may assume that $|\sigma| = 2$ on V_Q and only work with the sublattice $\bar{Q}$. For simplicity, we use Q instead of $\bar{Q}$ for the rest of this section.

3.2. **Restricting the orbifold V_Q^σ to V_L^σ.** By [FHL, LL], we have that the subalgebra V_L of V_Q is isomorphic to the tensor product $V_{L_+} \otimes V_{L_-}$, since $L = L_+ \oplus L_-$ is an orthogonal direct sum. Note that σ acts as the identity operator on L_+ and as -1 on L_-. Then $V_L^\sigma \cong V_{L_+} \otimes V_{L_-}^+$ is a subalgebra of V_Q^σ.

Proposition 3.5. *Every V_Q^σ-module is a direct sum of irreducible V_L^σ-modules. In particular, V_Q^σ has this form.*

Proof. It is shown in Theorem 3.16 of [DLM2] that the vertex algebra V_{L_+} is regular, since L_+ is positive definite. It is also shown in [A2, ABD, DJL] that the vertex algebra $V_{L_-}^+$ is regular. Since the tensor product of regular vertex algebras is again regular (Proposition 3.3 in [DLM2]), we have that $V_L^\sigma \cong V_{L_+} \otimes V_{L_-}^+$ is also regular. □

In order to obtain a precise description of V_Q^σ, we will decompose V_Q as a direct sum of irreducible modules over V_L^σ. This is done in two steps. The first step is to break V_Q as a direct sum of V_L-modules, using the cosets of Q modulo L. Since the lattice Q is integral, we have $Q \subseteq L^*$ and we can view Q/L as a subgroup of L^*/L. It follows that

$$V_Q = \bigoplus_{\gamma+L\in Q/L} V_{\gamma+L} \,,$$

where each $V_{\gamma+L}$ is an irreducible V_L-module [D1]. Writing $\gamma = \gamma_+ + \gamma_-$, we get

$$V_{\gamma+L} \cong V_{\gamma_++L_+} \otimes V_{\gamma_-+L_-}$$

as modules over $V_L \cong V_{L_+} \otimes V_{L_-}$. Therefore,

$$V_Q \cong \bigoplus_{\gamma+L\in Q/L} V_{\gamma_++L_+} \otimes V_{\gamma_-+L_-} \tag{3.5}$$

as V_L-modules. Note that $\gamma_+ + L_+$ and $\gamma_- + L_-$ depend only on the coset $\gamma + L$ and not on the representative $\gamma \in Q$, because $\alpha_\pm \in L_\pm$ for $\alpha \in L$.

Since $\sigma\gamma_- = -\gamma_-$ and $2\gamma_- \in L_-$, it follows that σ acts on the V_{L_-}-module $V_{\gamma_-+L_-}$. The second step is to decompose each module $V_{\gamma_-+L_-}$ into eigenspaces for σ, which are irreducible as $V_{L_-}^+$-modules [AD]. We thus obtain the following description of V_Q^σ.

Proposition 3.6. *The orbifold V_Q^σ decomposes as a direct sum of irreducible V_L^σ-modules as follows*:

$$V_Q^\sigma \cong \bigoplus_{\gamma+L\in Q/L} V_{\gamma_++L_+} \otimes V_{\gamma_-+L_-}^{\eta(\gamma)} \,. \tag{3.6}$$

Proof. Using (3.5), it is enough to determine the subspace S_γ of σ-invariants in $V_{\gamma_++L_+} \otimes V_{\gamma_-+L_-}$, for a fixed $\gamma \in Q$. As a V_L^σ-module, S_γ is generated by the element (cf. (2.16)):

$$v_\gamma = e^\gamma + \eta(\gamma) e^{\sigma\gamma} = e^{\gamma_+} \otimes (e^{\gamma_-} + \eta(\gamma) e^{-\gamma_-}) \in V_{\gamma_++L_+} \otimes V_{\gamma_-+L_-}^{\eta(\gamma)} \,. \tag{3.7}$$

Since $V_{\gamma_++L_+} \otimes V_{\gamma_-+L_-}^{\eta(\gamma)}$ is an irreducible V_L^σ-module, it must be equal to S_γ. ∎

From the study of tensor products in [FHL, LL], all irreducible modules over $V_L^\sigma \cong V_{L_+} \otimes V_{L_-}^+$ are tensor products of irreducible modules over the factors V_{L_+} and $V_{L_-}^+$. By the results of [D1, DN, AD] reviewed in Sections 2.5 and 2.6, we obtain that all irreducible V_L^σ-modules have the form:

(1) $V_{\lambda+L_+} \otimes V_{\mu+L_-}$, where $\lambda \in L_+^*$, $\mu \in L_-^*$, $2\mu \notin L_-$,
(2) $V_{\lambda+L_+} \otimes V_{\mu+L_-}^\pm$, where $\lambda \in L_+^*$, $\mu \in L_-^*$, $2\mu \in L_-$,
(3) $V_{\lambda+L_+} \otimes V_{L_-}^{T,\pm}$, where $\lambda \in L_+^*$,

and T is an irreducible module for the group G_σ associated to the lattice L_-. We refer to the V_L^σ-modules obtained from untwisted V_L-modules as *orbifold modules of untwisted type* and the ones obtained from twisted V_L-modules as *orbifold modules of twisted type.*

3.3. **Irreducible Modules over V_Q^σ.** In this subsection, we present our main result, the explicit classification of irreducible V_Q^σ-modules. As a consequence, we will find all of them as submodules of twisted or untwisted V_Q-modules. Recall that, by (3.4), we can assume that $Q = \bar{Q}$.

Theorem 3.7. *Let Q be a positive-definite even lattice, and σ be an automorphism of Q of order two such that $(\alpha|\sigma\alpha)$ is even for all $\alpha \in Q$. Then as a module over $V_L^\sigma \cong V_{L_+} \otimes V_{L_-}^+$ each irreducible V_Q^σ-module*

is isomorphic to one of the following:

$$\bigoplus_{\gamma+L\in Q/L} V_{\gamma_++\lambda+L_+}\otimes V_{\gamma_-+\mu+L_-} \qquad (2\mu\notin L_-)\,, \tag{3.8}$$

$$\bigoplus_{\gamma+L\in Q/L} V_{\gamma_++\lambda+L_+}\otimes V^{\epsilon\eta(\gamma)}_{\gamma_-+\mu+L_-} \qquad (2\mu\in L_-)\,, \tag{3.9}$$

$$\bigoplus_{\gamma+L\in Q/L} V_{\gamma_++\lambda+L_+}\otimes V^{T_\chi^{(\gamma_-)},\epsilon_\gamma}_{L_-}\,, \tag{3.10}$$

where $\lambda\in L_+^*$, $\mu\in L_-^*$, $\epsilon\in\{\pm\}$, χ *is a central character for the group* G_σ *associated to* L_-, *and* $\epsilon_\gamma=\epsilon\eta(\gamma)c_\chi(\gamma_-)$.

Proof. Let W be an irreducible V_Q^σ-module. Then W is a V_L^σ-module by restriction and, by Proposition 3.5, W is a direct sum of irreducible V_L^σ-modules. Suppose $A\subseteq W$ is an irreducible V_L^σ-module, and define $A^{(\gamma)}$ from A as follows. If

$$A=V_{\lambda+L_+}\otimes V_{\mu+L_-},\quad V_{\lambda+L_+}\otimes V^{\epsilon}_{\mu+L_-},\quad \text{or}\quad V_{\lambda+L_+}\otimes V^{T_\chi,\epsilon}_{L_-},$$

then

$$A^{(\gamma)}=V_{\lambda+\gamma_++L_+}\otimes V_{\mu+\gamma_-+L_-},\quad V_{\lambda+\gamma_++L_+}\otimes V^{\epsilon\eta(\gamma)}_{\mu+\gamma_-+L_-},$$
$$\text{and}\quad V_{\lambda+\gamma_++L_+}\otimes V^{T_\chi^{(\gamma_-)},\epsilon_\gamma}_{L_-},$$

respectively, where $\epsilon\in\{\pm\}$ and $\epsilon_\gamma=\epsilon\eta(\gamma)c_\chi(\gamma_-)$. We will consider separately the untwisted and twisted types.

Let A be of untwisted type, i.e., one of the modules $V_{\lambda+L_+}\otimes V_{\mu+L_-}$ for $2\mu\notin L_-$, or $V_{\lambda+L_+}\otimes V^{\epsilon}_{\mu+L_-}$ for $2\mu\in L_-$. Let $B\subseteq W$ be another irreducible V_L^σ-module that is possibly of twisted type. By Proposition 3.6, V_Q^σ is a direct sum of irreducible V_L^σ-modules generated by the vectors v_γ from (3.7), where $\gamma\in Q$. By restricting the field $Y(v_\gamma,z)$ to A and then projecting onto B, we obtain an intertwining operator of V_L^σ-modules of type $\binom{B}{V^{\eta(\gamma)}_{\gamma+L}\ \ A}$. From the study of intertwining operators in [ADL], we have that the intertwining operator $Y(v_\gamma,z)$ can be written as the tensor product

$$Y(v_\gamma,z)=Y(e^{\gamma_+},z)\otimes Y(e^{\gamma_-}+\eta(\gamma)e^{-\gamma_-},z),$$

where $Y(e^{\gamma_+},z)$ is an intertwining operator of type $\binom{V_{\lambda'+L_+}}{V_{\gamma_++L_+}\ \ V_{\lambda+L_+}}$ and $Y(e^{\gamma_-}+\eta(\gamma)e^{-\gamma_-},z)$ is an intertwining operator of type $\binom{V_{\mu'+L_-}}{V^{\eta(\gamma)}_{\gamma_-+L_-}\ \ V_{\mu+L_-}}$ or of type $\binom{V^{\pm\eta(\gamma)}_{\mu'+L_-}}{V^{\eta(\gamma)}_{\gamma_-+L_-}\ \ V^{\pm}_{\mu+L_-}}$. By [DL], the fusion

rules for $Y(e^{\gamma_+},z)$ are zero unless $\lambda' = \lambda+\gamma_+$. Since $\gamma_- \in L_-^* \cap \frac{1}{2}L_-$ and $|\gamma_-|^2 \in \mathbb{Z}$, we have that $\pi_{\gamma_-,2\mu} = 1$ when $2\mu \in L_-$ (cf. (2.22)). Hence the fusion rules for $Y(e^{\gamma_-}+\eta(\gamma)e^{-\gamma_-},z)$ are zero unless $\mu' = \mu+\gamma_-$, by Theorem 2.7. Therefore, for $\gamma + L \in Q/L$, we have $B = A^{(\gamma)}$. Hence $A \subseteq W$ implies that $\bigoplus_{\gamma\in Q/L} A^{(\gamma)} \subseteq W$. Since W is irreducible, we obtain that

$$W = \bigoplus_{\gamma+L\in Q/L} A^{(\gamma)}.$$

Now let $A = V_{\lambda+L_+} \otimes V_{L_-}^{T_\chi,\pm}$ and $B \subseteq W$ be another irreducible V_L^σ-module that is possibly of untwisted type. As above, the field $Y(v_\gamma,z)$ gives rise to an intertwining operator of V_L^σ-modules of type $\begin{pmatrix} & B & \\ V_{\gamma+L}^{\eta(\gamma)} & & A \end{pmatrix}$ and can be written as the tensor product

$$Y(v_\gamma,z) = Y(e^{\gamma_+},z) \otimes Y(e^{\gamma_-}+\eta(\gamma)e^{-\gamma_-},z),$$

where $Y(e^{\gamma_+},z)$ is an intertwining operator of type $\begin{pmatrix} & V_{\lambda'+L_+} & \\ V_{\gamma_++L_+} & & V_{\lambda+L_+} \end{pmatrix}$ and $Y(e^{\gamma_-}+\eta(\gamma)e^{-\gamma_-},z)$ is an intertwining operator of type $\begin{pmatrix} & V_{L_-}^{T_{\chi'},\pm\epsilon} & \\ V_{\gamma_-+L_-}^{\eta(\gamma)} & & V_{L_-}^{T_\chi,\pm} \end{pmatrix}$, where $\epsilon = c_\chi(\gamma)\eta(\gamma)$. As with the untwisted type, the fusion rules for $Y(e^{\gamma_+},z)$ are zero unless $\lambda' = \lambda+\gamma_+$. By Theorem 2.7, the action of $Y(e^{\gamma_-},z)$ on $V_{L_-}^{T_\chi}$ is determined by computing $c_\chi(\gamma_-)$ (cf. (2.23)) and is zero unless $\chi' = \chi^{(\gamma_-)}$. Since the lattice $\mathbb{Z}\gamma_- + L_-$ is integral (by Corollary 3.4), the map ε can be extended to this lattice with values ±1. Therefore $\varepsilon(\gamma_-,2\gamma_-) = \varepsilon(\gamma_-,\gamma_-)^2 = 1$ and (2.23) becomes $c_\chi(\gamma_-) = s(2\gamma_-)$; see (2.19). Hence the eigenspace of each summand in the V_Q^σ-module may change depending on the signs of each $U_{2\gamma_-}$. Therefore, $B = A^{(\gamma)}$ and

$$W = \bigoplus_{\gamma+L\in Q/L} A^{(\gamma)}.$$

This completes the proof. $\square$

Theorem 3.8. *Let Q be a positive-definite even lattice, and σ be an automorphism of Q of order two such that $(\alpha|\sigma\alpha)$ is even for all $\alpha \in Q$. Then every irreducible V_Q^σ-module is a submodule of a V_Q-module or a σ-twisted V_Q-module.*

Proof. Let us consider first the untwisted case. By Theorem 3.7, any irreducible V_Q^σ-module W of untwisted type is given by (3.8) or (3.9) when considered as a V_L^σ-module by restriction. For a fixed $\gamma \in Q$, the

nonzero fusion rules for $Y(e^{\gamma_+}, z)$ and $Y(e^{\gamma_-} + \eta(\gamma)e^{-\gamma_-}, z)$ are equal to 1 and the intertwining operators in [ADL] are given by the usual formula (2.11) up to a scalar multiple. Using that

$$(\gamma|\lambda+\mu) - (\sigma\gamma|\lambda+\mu) = (2\gamma_-|\lambda+\mu) = (2\gamma_-|\mu) \in \mathbb{Z}\,,$$

we have that for some $m \in \mathbb{Z}$,

$$\begin{aligned} Y(v_\gamma, z)e^{\lambda+\mu} &= Y(e^\gamma + \eta(\gamma)e^{\sigma\gamma}, z)e^{\lambda+\mu} \\ &= z^{(\gamma|\lambda+\mu)}\big(E(\gamma,z)e^{\gamma+\lambda+\mu} + \eta(\gamma)z^m E(\sigma\gamma,z)e^{\sigma\gamma+\lambda+\mu}\big)\,, \end{aligned}$$

where

$$E(\alpha,z) = \exp\Big(\sum_{n<0}\alpha_n\frac{z^{-n}}{-n}\Big)\exp\Big(\sum_{n>0}\alpha_n\frac{z^{-n}}{-n}\Big)$$

contains only integral powers of z. This implies that $(\lambda+\mu|\gamma) \in \mathbb{Z}$ for all $\gamma \in Q$, i.e., $\lambda+\mu \in Q^*$. Then for the untwisted V_Q-module $V_{\lambda+\mu+Q}$, we have that

$$\begin{aligned} V_{\lambda+\mu+Q} &= \bigoplus_{\gamma+L\in Q/L} V_{\gamma+\lambda+\mu+L} \\ &\cong \bigoplus_{\gamma+L\in Q/L} V_{\gamma_++\lambda+L_+}\otimes V_{\gamma_-+\mu+L_-} \end{aligned}$$

as a direct sum of irreducible V_L-modules. Using the intertwining operators in [ADL], we see that W is a submodule of the restriction of $V_{\lambda+\mu+Q}$ to V_Q^σ.

Now we consider the twisted case. By Theorem 3.7, any irreducible V_Q^σ-module W of twisted type is given by (3.10) when considered as a V_L^σ-module by restriction. Then, for $\gamma \in Q$, the nonzero fusion rules for $Y(e^{\gamma_+}, z)$ and $Y(e^{\gamma_-} + \eta(\gamma)e^{-\gamma_-}, z)$ are equal to 1 and the intertwining operators in [ADL] are given by the usual formula (2.17) up to a scalar multiple. Since these scalars can be absorbed in U_γ, we will have (2.17) without loss of generality. Therefore, the action of $Y(e^\gamma, z)$ on W can be determined, so that its modes are linear maps

$$V_{\lambda+L_+}\otimes V_{L_-}^{T_\chi,\epsilon} \to V_{\gamma_++\lambda+L_+}\otimes V_{L_-}^{T_\chi^{(\gamma_-)},\epsilon_\gamma} \qquad (\epsilon\in\{\pm\})\,,$$

where $\epsilon_\gamma = \epsilon c_\chi(\gamma)\eta(\gamma)$ (cf. (3.10)). Hence the σ-twisted V_Q-module is given by

$$\bigoplus_{\gamma+L\in Q/L} V_{\gamma_++\lambda+L_+}\otimes V_{L_-}^{T_\chi^{(\gamma_-)}}$$

and its restriction to V_Q^σ contains W. $\square$

4. Root lattices and Dynkin diagram automorphisms

In this section, we present examples of the lattice Q being a root lattice of type ADE, corresponding to the simply-laced simple Lie algebras. We use the classification from [D1, DN, AD] and the construction of twisted modules from Section 2.5 to construct explicitly all irreducible V_Q^σ-modules. In each case, a correspondence between the two constructions is shown. In order to apply Theorems 3.7 and 3.8, we first calculate $\bar{Q}$ and L. Then the twisted V_L-modules are found. When necessary, the intertwiners from [ADL] are used to construct the V_Q^σ-modules. The untwisted and twisted $V_{\bar{Q}}$-modules are calculated using $(\bar{Q})^*/\bar{Q}$ and its σ-invariant elements. For more details and additional examples, the reader is referred to [E].

4.1. A_2 **root lattice with a Dynkin diagram automorphism.** Consider the A_2 simple roots $\{\alpha_1, \alpha_2\}$, the associated root lattice $Q = \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2$, and the Dynkin diagram automorphism $\sigma\colon \alpha_1 \leftrightarrow \alpha_2$. The case of A_n root lattice for even n is similar and is treated in [E].

Set $\alpha = \alpha_1 + \alpha_2$ and $\beta = \alpha_1 - \alpha_2$. Then

$$|\alpha|^2 = 2\,, \qquad |\beta|^2 = 6\,, \qquad (\alpha|\beta) = 0\,,$$

and

$$L_+ = \mathbb{Z}\alpha\,, \qquad L_- = \mathbb{Z}\beta\,, \qquad \bar{Q} = L = L_+ \oplus L_-\,.$$

Therefore,

$$V_Q^\sigma = V_{\bar{Q}}^\sigma = V_L^\sigma \cong V_{\mathbb{Z}\alpha} \otimes V_{\mathbb{Z}\beta}^+\,.$$

Using the results of [FHL, D1, DN], we obtain a total of 20 distinct irreducible V_L^σ-modules:

$$\begin{array}{ll}
V_{\mathbb{Z}\alpha} \otimes V_{\mathbb{Z}\beta}^\pm\,, & V_{\frac{\alpha}{2}+\mathbb{Z}\alpha} \otimes V_{\mathbb{Z}\beta}^\pm\,, \\
V_{\mathbb{Z}\alpha} \otimes V_{\frac{\beta}{2}+\mathbb{Z}\beta}^\pm\,, & V_{\frac{\alpha}{2}+\mathbb{Z}\alpha} \otimes V_{\frac{\beta}{2}+\mathbb{Z}\beta}^\pm\,, \\
V_{\mathbb{Z}\alpha} \otimes V_{\frac{\beta}{6}+\mathbb{Z}\beta}\,, & V_{\frac{\alpha}{2}+\mathbb{Z}\alpha} \otimes V_{\frac{\beta}{6}+\mathbb{Z}\beta}\,, \\
V_{\mathbb{Z}\alpha} \otimes V_{\frac{\beta}{3}+\mathbb{Z}\beta}\,, & V_{\frac{\alpha}{2}+\mathbb{Z}\alpha} \otimes V_{\frac{\beta}{3}+\mathbb{Z}\beta}\,, \\
V_{\mathbb{Z}\alpha} \otimes V_{\mathbb{Z}\beta}^{T_j,\pm}\,, & V_{\frac{\alpha}{2}+\mathbb{Z}\alpha} \otimes V_{\mathbb{Z}\beta}^{T_j,\pm} \qquad (j = 1, 2)\,,
\end{array}$$

where T_1, T_2 denote the two irreducible modules over the group G_σ associated to the lattice $\mathbb{Z}\beta$. They are 1-dimensional and on them $U_\beta = \pm\mathrm{i}/64$ by (2.19).

On the other hand, by Dong's Theorem [D1], the irreducible V_L-modules have the form $V_{\lambda+L}$ $(\lambda + L \in L^*/L)$. If $\sigma(\lambda + L) = \lambda + L$, the module $V_{\lambda+L}$ breaks into eigenspaces $V_{\lambda+L}^\pm$ of σ; otherwise, σ is

an isomorphism of V_L^σ-modules $V_{\lambda+L} \to V_{\sigma(\lambda)+L}$. Thus, there are 12 distinct irreducible V_L^σ-modules of untwisted type:

$$V_L^{\pm}\,,\quad V^{\pm}_{\frac{\alpha}{2}+L}\,,\quad V^{\pm}_{\frac{\beta}{2}+L}\,,\quad V^{\pm}_{\frac{\alpha}{2}+\frac{\beta}{2}+L}\,,$$
$$V_{\frac{\beta}{6}+L}\,,\quad V_{\frac{\beta}{3}+L}\,,\quad V_{\frac{\alpha}{2}+\frac{\beta}{6}+L}\,,\quad V_{\frac{\alpha}{2}+\frac{\beta}{3}+L}\,.$$

The correspondence of these modules to those above is given by:

$$V_{m\alpha+n\beta+L} \cong V_{m\alpha+L_+} \otimes V_{n\beta+L_-} \qquad (m,n\in\mathbb{Q})\,.$$

Now we will construct the σ-twisted V_L-modules using Section 2.5. Consider the irreducible modules over the group G_σ associated to the lattice L. One finds that on them $U_\beta = \pm\mathrm{i}/64$ and U_α acts freely. Hence, such modules can be identified with the space $P = \mathbb{C}[q,q^{-1}]$, where U_α acts as a multiplication by q. Then on P we have:

$$e^{\pi\mathrm{i}\alpha_{(0)}} = s_1\,,\qquad U_\alpha = q\,,\qquad U_\beta = s_2\frac{\mathrm{i}}{64}\,,$$

where the signs $s_1, s_2 \in \{\pm\}$ are independent. The corresponding four G_σ-modules will be denoted as $P_{(s_1,s_2)}$. The irreducible σ-twisted V_L-modules have the form $\mathcal{F}_\sigma \otimes P_{(s_1,s_2)}$, where $\mathcal{F}_\sigma$ is the σ-twisted Fock space (see (2.12)). Next, we restrict the σ-twisted V_L-modules to V_L^σ. The automorphism σ acts on each $P_{(s_1,s_2)}$ as the identity operator, since

$$\sigma(q^n) = \sigma(U_\alpha^n\cdot 1) = U_\alpha^n \cdot 1 = q^n\,.$$

We obtain 8 irreducible V_L^σ-modules of twisted type:

$$\mathcal{F}_\sigma^{\pm} \otimes P_{(s_1,s_2)}\,,\qquad s_1,s_2\in\{\pm\}\,,$$

where $\mathcal{F}_\sigma^{\pm}$ are the ±1-eigenspaces of σ. We have the following correspondence among irreducible V_L^σ-modules of twisted type:

$$\mathcal{F}_\sigma^{\pm}\otimes P_{(+,+)} \cong V_{\mathbb{Z}\alpha}\otimes V_{\mathbb{Z}\beta}^{T_1,\pm}, \qquad \mathcal{F}_\sigma^{\pm}\otimes P_{(-,+)} \cong V_{\frac{\alpha}{2}+\mathbb{Z}\alpha}\otimes V_{\mathbb{Z}\beta}^{T_1,\pm},$$
$$\mathcal{F}_\sigma^{\pm}\otimes P_{(+,-)} \cong V_{\mathbb{Z}\alpha}\otimes V_{\mathbb{Z}\beta}^{T_2,\pm}, \qquad \mathcal{F}_\sigma^{\pm}\otimes P_{(-,-)} \cong V_{\frac{\alpha}{2}+\mathbb{Z}\alpha}\otimes V_{\mathbb{Z}\beta}^{T_2,\pm}.$$

4.2. A_2 **root lattice with the negative of a Dynkin diagram automorphism.** Consider now the negative Dynkin diagram automorphism $\phi = -\sigma$. Keeping the notation from the previous subsection, we have: $L_+ = \mathbb{Z}\beta$, $L_- = \mathbb{Z}\alpha$, and

$$V_Q^\phi \cong V_{\mathbb{Z}\beta}\otimes V_{\mathbb{Z}\alpha}^{+}\,.$$

Furthermore, $L_+^*/L_+ = \mathbb{Z}\frac{\beta}{6}/\mathbb{Z}\beta$ and $\left(L_-^*/L_-\right)^\phi = L_-^*/L_- = \mathbb{Z}\frac{\alpha}{2}/\mathbb{Z}\alpha$. Thus, using the results of [FHL, D1, DN], we obtain a total of 48 distinct irreducible V_Q^ϕ-modules:

$$V_{i\frac{\beta}{6}+\mathbb{Z}\beta}\otimes V_{\mathbb{Z}\alpha}^{\pm}\,,\qquad V_{i\frac{\beta}{6}+\mathbb{Z}\beta}\otimes V^{\pm}_{\frac{\alpha}{2}+\mathbb{Z}\alpha}\,,\qquad V_{i\frac{\beta}{6}+\mathbb{Z}\beta}\otimes V_{\mathbb{Z}\alpha}^{T_j,\pm}\,,$$

where $i = 0, \dots, 5$ and $j = 1, 2$.

4.3. A_3 **root lattice with a Dynkin diagram automorphism.** Consider the A_3 simple roots $\{\alpha_1, \alpha_2, \alpha_3\}$, the root lattice $Q = \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2 + \mathbb{Z}\alpha_3$, and the Dynkin diagram automorphism $\sigma\colon \alpha_1 \leftrightarrow \alpha_3$, $\alpha_2 \leftrightarrow \alpha_2$. The case of A_n root lattice for odd n is similar and is treated in [E].

Set $\alpha = \alpha_1 + \alpha_3$ and $\beta = \alpha_1 - \alpha_3$. Then $\bar{Q} = Q$ and

$$L_+ = \mathbb{Z}\alpha + \mathbb{Z}\alpha_2 \,, \qquad L_- = \mathbb{Z}\beta \,, \qquad Q/L = \{L, \alpha_1 + L\} \,.$$

Hence, by Proposition 3.6,

$$V_Q^\sigma \cong \big(V_{L_+} \otimes V_{\mathbb{Z}\beta}^+\big) \oplus \big(V_{\frac{\alpha}{2}+L_+} \otimes V^+_{\frac{\beta}{2}+\mathbb{Z}\beta}\big).$$

It follows from Theorems 2.7 and 3.7 that the irreducible V_Q^σ-modules are given by:

$$\begin{aligned}
&\big(V_{L_+} \otimes V_{\mathbb{Z}\beta}^\pm\big) \oplus \big(V_{\frac{\alpha}{2}+L_+} \otimes V^\pm_{\frac{\beta}{2}+\mathbb{Z}\beta}\big) \,,\\
&\big(V_{L_+} \otimes V^\pm_{\frac{\beta}{2}+\mathbb{Z}\beta}\big) \oplus \big(V_{\frac{\alpha}{2}+L_+} \otimes V_{\mathbb{Z}\beta}^\pm\big) \,,\\
&\big(V_{\frac{\alpha_2}{2}+L_+} \otimes V_{\frac{\beta}{4}+\mathbb{Z}\beta}\big) \oplus \big(V_{\frac{\alpha_2}{2}+\frac{\alpha}{2}+L_+} \otimes V_{\frac{\beta}{4}+\mathbb{Z}\beta}\big)\\
&\big(V_{L_+} \otimes V_{\mathbb{Z}\beta}^{T_1,\pm}\big) \oplus \big(V_{\frac{\alpha}{2}+L_+} \otimes V_{\mathbb{Z}\beta}^{T_1,\pm}\big) \,,\\
&\big(V_{L_+} \otimes V_{\mathbb{Z}\beta}^{T_2,\pm}\big) \oplus \big(V_{\frac{\alpha}{2}+L_+} \otimes V_{\mathbb{Z}\beta}^{T_2,\mp}\big) \,,
\end{aligned}$$

where $U_\beta = (-1)^j/16$ on T_j for $j = 1, 2$ (see (2.19)).

On the other hand, by Dong's Theorem [D1], the irreducible V_Q-modules have the form $V_{\lambda+Q}$ $(\lambda + Q \in Q^*/Q)$. We have

$$Q^*/Q = \{Q, \lambda_1 + Q, 2\lambda_1 + Q, 3\lambda_1 + Q\} \,,$$

where

$$\lambda_1 = \frac{1}{4}(3\alpha_1 + 2\alpha_2 + \alpha_3) \,.$$

Note that Q and $2\lambda_1 + Q = \frac{\alpha}{2} + Q$ are σ-invariant, while $\sigma(\lambda_1 + Q) = 3\lambda_1 + Q$. Thus, there are 5 distinct irreducible V_Q^σ-modules of untwisted type:

$$V_Q^\pm \,, \qquad V^\pm_{\frac{\alpha}{2}+Q} \,, \qquad V_{\lambda_1+Q} \,.$$

We have the following correspondence:

$$\begin{aligned}
V_Q^\pm &\cong \big(V_{L_+} \otimes V_{\mathbb{Z}\beta}^\pm\big) \oplus \big(V_{\frac{\alpha}{2}+L_+} \otimes V^\pm_{\frac{\beta}{2}+\mathbb{Z}\beta}\big) \,,\\
V^\pm_{\frac{\alpha}{2}+Q} &\cong \big(V_{L_+} \otimes V^\pm_{\frac{\beta}{2}+\mathbb{Z}\beta}\big) \oplus \big(V_{\frac{\alpha}{2}+L_+} \otimes V_{\mathbb{Z}\beta}^\pm\big) \,,\\
V_{\lambda_1+Q} &\cong \big(V_{\frac{\alpha_2}{2}+L_+} \otimes V_{\frac{\beta}{4}+\mathbb{Z}\beta}\big) \oplus \big(V_{\frac{\alpha_2}{2}+\frac{\alpha}{2}+L_+} \otimes V_{\frac{\beta}{4}+\mathbb{Z}\beta}\big) \,.
\end{aligned}$$

We now construct the σ-twisted V_Q-modules using Section 2.5. Consider the irreducible modules over the group G_σ associated to the lattice Q. We find from (2.19) that on them $U_\beta = \pm 1/16$, and there are two such modules, P_1 and P_2, corresponding to $U_\beta = -1/16$ and $U_\beta = 1/16$, respectively. For both $j = 1, 2$, we can identify $P_j = \mathbb{C}[q, q^{-1}, p, p^{-1}]$ so that

$$U_{\alpha_1} = q\,, \qquad U_{\alpha_2} = p(-1)^{q\frac{\partial}{\partial q}}\,.$$

Then on P_j we have:

$$U_{\alpha_3} = (-1)^{j+1} q\,, \qquad U_\alpha = (-1)^{j+1} 4q^2\,, \qquad U_\beta = \frac{(-1)^j}{16}\,.$$

The automorphism σ acts on each of these modules: σ is the identity operator on P_1, while on P_2 we have $\sigma = (-1)^{q\frac{\partial}{\partial q}}$. Hence, P_2 decomposes into two eigenspaces $P_2^\pm$ with eigenvalues ± 1. The irreducible σ-twisted V_Q-modules have the form $\mathcal{F}_\sigma \otimes P_j$ for $j = 1, 2$, where $\mathcal{F}_\sigma$ is the σ-twisted Fock space (see (2.12)). Since $\mathcal{F}_\sigma$ itself decomposes into ± 1-eigenspaces of σ, we obtain 4 distinct irreducible V_Q^σ-modules of twisted type:

$$\begin{aligned}
\bigl(\mathcal{F}_\sigma \otimes P_1\bigr)^\pm &= \mathcal{F}_\sigma^\pm \otimes P_1\,, \\
\bigl(\mathcal{F}_\sigma \otimes P_2\bigr)^\pm &= \bigl(\mathcal{F}_\sigma^\pm \otimes P_2^+\bigr) \oplus \bigl(\mathcal{F}_\sigma^\mp \otimes P_2^-\bigr)\,.
\end{aligned}$$

We have the following correspondence:

$$\begin{aligned}
\bigl(\mathcal{F}_\sigma \otimes P_1\bigr)^\pm &\cong \bigl(V_{L_+} \otimes V_{\mathbb{Z}\beta}^{T_1,\pm}\bigr) \oplus \bigl(V_{\frac{\alpha}{2}+L_+} \otimes V_{\mathbb{Z}\beta}^{T_1,\pm}\bigr)\,, \\
\bigl(\mathcal{F}_\sigma \otimes P_2\bigr)^\pm &\cong \bigl(V_{L_+} \otimes V_{\mathbb{Z}\beta}^{T_2,\pm}\bigr) \oplus \bigl(V_{\frac{\alpha}{2}+L_+} \otimes V_{\mathbb{Z}\beta}^{T_2,\mp}\bigr)\,.
\end{aligned}$$

4.4. A_3 **root lattice with the negative of a Dynkin diagram automorphism.** Consider now the negative Dynkin diagram automorphism, $\phi = -\sigma$. With the notation from the previous subsection, we have:

$$L_+ = \mathbb{Z}\beta\,, \qquad L_- = \mathbb{Z}\alpha + \mathbb{Z}\alpha_2\,, \qquad \bigl(L_-^*/L_-\bigr)^\phi = L_-^*/L_-\,,$$

and

$$V_Q^\phi \cong \bigl(V_{\mathbb{Z}\beta} \otimes V_{L_-}^+\bigr) \oplus \bigl(V_{\frac{\beta}{2}+\mathbb{Z}\beta} \otimes V_{\frac{\alpha}{2}+L_-}^+\bigr)\,.$$

By Theorem 3.7, the irreducible V_Q^ϕ-modules of untwisted type are given by:

$$\begin{aligned}
&\big(V_{\mathbb{Z}\beta}\otimes V^{\pm}_{L_-}\big)\oplus\big(V_{\frac{\beta}{2}+\mathbb{Z}\beta}\otimes V^{\pm}_{\frac{\alpha}{2}+L_-}\big)\,,\\
&\big(V_{\mathbb{Z}\beta}\otimes V^{\pm}_{\frac{\alpha}{2}+L_-}\big)\oplus\big(V_{\frac{\beta}{2}+\mathbb{Z}\beta}\otimes V^{\pm}_{L_-}\big)\,,\\
&\big(V_{\frac{\beta}{4}+\mathbb{Z}\beta}\otimes V^{\pm}_{\frac{\alpha_2}{2}+L_-}\big)\oplus\big(V_{\frac{3\beta}{4}+\mathbb{Z}\beta}\otimes V^{\pm}_{\frac{\alpha+\alpha_2}{2}+L_-}\big)\,,\\
&\big(V_{\frac{\beta}{4}+\mathbb{Z}\beta}\otimes V^{\pm}_{\frac{\alpha+\alpha_2}{2}+L_-}\big)\oplus\big(V_{\frac{3\beta}{4}+\mathbb{Z}\beta}\otimes V^{\pm}_{\frac{\alpha_2}{2}+L_-}\big)\,,
\end{aligned}$$

and the ones of twisted type are:

$$\begin{aligned}
&\big(V_{\mathbb{Z}\beta}\otimes V^{T_1,\pm}_{L_-}\big)\oplus\big(V_{\frac{\beta}{2}+\mathbb{Z}\beta}\otimes V^{T_1,\pm}_{L_-}\big)\,,\\
&\big(V_{\mathbb{Z}\beta}\otimes V^{T_2,\pm}_{L_-}\big)\oplus\big(V_{\frac{\beta}{2}+\mathbb{Z}\beta}\otimes V^{T_2,\mp}_{L_-}\big)\,.
\end{aligned}$$

4.5. D_n **root lattice with a Dynkin diagram automorphism.** Consider the D_n simple roots $\{\alpha_1,\dots,\alpha_n\}$, where $n\geq 4$, the root lattice Q, and the Dynkin diagram automorphism $\sigma\colon \alpha_{n-1}\leftrightarrow\alpha_n$ and $\alpha_i\leftrightarrow\alpha_i$ for $i=1,\dots,n-2$. This case is similar to the A_3 case discussed in Section 4.3, so we will be brief (see [E] for details).

Set $\alpha=\alpha_{n-1}+\alpha_n$ and $\beta=\alpha_{n-1}-\alpha_n$. Then

$$L_+=\mathbb{Z}\alpha+\sum_{i=1}^{n-2}\mathbb{Z}\alpha_i\,,\qquad L_-=\mathbb{Z}\beta\,,\qquad Q=\bar{Q}\,,$$

and

$$Q/L=\{L,\alpha_{n-1}+L\}\,.$$

Hence, by Proposition 3.6,

$$V_Q^\sigma\cong\big(V_{L_+}\otimes V^{+}_{\mathbb{Z}\beta}\big)\oplus\big(V_{\frac{\alpha}{2}+L_+}\otimes V^{+}_{\frac{\beta}{2}+\mathbb{Z}\beta}\big)\,.$$

By Theorems 2.7 and 3.7, the irreducible V_Q^σ-modules are given by:

$$\begin{aligned}
&\big(V_{L_+}\otimes V^{\pm}_{\mathbb{Z}\beta}\big)\oplus\big(V_{\frac{\alpha}{2}+L_+}\otimes V^{\pm}_{\frac{\beta}{2}+\mathbb{Z}\beta}\big)\,,\\
&\big(V_{L_+}\otimes V^{\pm}_{\frac{\beta}{2}+\mathbb{Z}\beta}\big)\oplus\big(V_{\frac{\alpha}{2}+L_+}\otimes V^{\pm}_{\mathbb{Z}\beta}\big)\,,\\
&\big(V_{\frac{n-1}{4}\alpha+\theta+L_+}\otimes V_{\frac{\beta}{4}+\mathbb{Z}\beta}\big)\oplus\big(V_{\frac{n+1}{4}\alpha+\theta+L_+}\otimes V_{\frac{\beta}{4}+\mathbb{Z}\beta}\big)\,,\\
&\big(V_{L_+}\otimes V^{T_1,\pm}_{\mathbb{Z}\beta}\big)\oplus\big(V_{\frac{\alpha}{2}+L_+}\otimes V^{T_1,\pm}_{\mathbb{Z}\beta}\big)\,,\\
&\big(V_{L_+}\otimes V^{T_2,\pm}_{\mathbb{Z}\beta}\big)\oplus\big(V_{\frac{\alpha}{2}+L_+}\otimes V^{T_2,\mp}_{\mathbb{Z}\beta}\big)\,,
\end{aligned}$$

where $\theta=\frac{1}{2}\sum_{i=0}^{\lfloor\frac{n-3}{2}\rfloor}\alpha_{2i+1}$ and $U_\beta=(-1)^j/16$ on T_j for $j=1,2$.

On the other hand, note that

$$Q^*/Q=\Big\{Q,\frac{\alpha}{2}+Q,\lambda_{n-1}+Q,\lambda_n+Q\Big\}\,,$$

where

$$\lambda_{n-1} = \frac{1}{2}\Big(\alpha_1 + 2\alpha_2 + \cdots + (n-2)\alpha_{n-2} + \frac{1}{2}n\alpha_{n-1} + \frac{1}{2}(n-2)\alpha_n\Big)$$

and $\lambda_n = \sigma(\lambda_{n-1})$. Using Dong's Theorem [D1], we obtain 5 distinct irreducible V_Q^σ-modules of untwisted type:

$$V_Q^{\pm}\,, \qquad V_{\frac{\alpha}{2}+Q}^{\pm}\,, \qquad V_{\lambda_{n-1}+Q}\,.$$

We have the following correspondence:

$$\begin{aligned}
V_Q^{\pm} &\cong \big(V_{L_+} \otimes V_{\mathbb{Z}\beta}^{\pm}\big) \oplus \big(V_{\frac{\alpha}{2}+L_+} \otimes V_{\frac{\beta}{2}+\mathbb{Z}\beta}^{\pm}\big)\,,\\
V_{\frac{\alpha}{2}+Q}^{\pm} &\cong \big(V_{L_+} \otimes V_{\frac{\beta}{2}+\mathbb{Z}\beta}^{\pm}\big) \oplus \big(V_{\frac{\alpha}{2}+L_+} \otimes V_{\mathbb{Z}\beta}^{\pm}\big)\,,\\
V_{\lambda_{n-1}+Q} &\cong \big(V_{\frac{n-1}{4}\alpha+\theta+L_+} \otimes V_{\frac{\beta}{4}+\mathbb{Z}\beta}\big) \oplus \big(V_{\frac{n+1}{4}\alpha+\theta+L_+} \otimes V_{\frac{\beta}{4}+\mathbb{Z}\beta}\big)\,.
\end{aligned}$$

We now construct the irreducible σ-twisted V_Q-modules using Section 2.5. There are two irreducible G_σ-modules, P_1 and P_2, and both can be identified with $\mathbb{C}[p_1^{\pm 1}, \dots, p_{n-1}^{\pm 1}]$ as vector spaces. The action of G_σ on P_j is determined by:

$$\begin{aligned}
U_{\alpha_i} &= p_i(-1)^{p_{i+1}\frac{\partial}{\partial p_{i+1}}}\,, \qquad i = 1, \dots, n-2\,,\\
U_{\alpha_{n-1}} &= p_{n-1}\,, \qquad U_{\alpha_n} = (-1)^{j+1}p_{n-1}\,,\\
U_\alpha &= (-1)^{j+1}4p_{n-1}^2\,, \qquad U_\beta = \frac{(-1)^j}{16}\,.
\end{aligned}$$

The automorphism σ acts as the identity operator on P_1, and as $(-1)^{q\frac{\partial}{\partial q}}$ on P_2. We obtain 4 distinct irreducible V_Q^σ-modules of twisted type:

$$\mathcal{F}_\sigma^{\pm} \otimes P_1\,, \qquad \big(\mathcal{F}_\sigma \otimes P_2\big)^{\pm} = \big(\mathcal{F}_\sigma^{\pm} \otimes P_2^{+}\big) \oplus \big(\mathcal{F}_\sigma^{\mp} \otimes P_2^{-}\big)\,,$$

and we have the following correspondence:

$$\begin{aligned}
\big(\mathcal{F}_\sigma \otimes P_1\big)^{\pm} &\cong \big(V_{L_+} \otimes V_{\mathbb{Z}\beta}^{T_1,\pm}\big) \oplus \big(V_{\frac{\alpha}{2}+L_+} \otimes V_{\mathbb{Z}\beta}^{T_1,\pm}\big)\,,\\
\big(\mathcal{F}_\sigma \otimes P_2\big)^{\pm} &\cong \big(V_{L_+} \otimes V_{\mathbb{Z}\beta}^{T_2,\pm}\big) \oplus \big(V_{\frac{\alpha}{2}+L_+} \otimes V_{\mathbb{Z}\beta}^{T_2,\mp}\big)\,.
\end{aligned}$$

4.6. E_6 **root lattice with a Dynkin diagram automorphism.** Consider the E_6 Dynkin diagram with the simple roots $\{\alpha_1, \dots, \alpha_6\}$ labeled as follows:

$$\begin{array}{ccccccccc}
 & & & & \alpha_6 & & & & \\
 & & & & | & & & & \\
\alpha_1 & \text{———} & \alpha_2 & \text{———} & \alpha_3 & \text{———} & \alpha_4 & \text{———} & \alpha_5
\end{array}$$

Let Q be the root lattice, and σ be the Dynkin diagram automorphism $\alpha_1 \leftrightarrow \alpha_5$, $\alpha_2 \leftrightarrow \alpha_4$, with fixed points α_3 and α_6. Set

$$\alpha^1 = \alpha_1 + \alpha_5\,, \quad \beta^1 = \alpha_1 - \alpha_5\,, \quad \alpha^2 = \alpha_2 + \alpha_4\,, \quad \beta^2 = \alpha_2 - \alpha_4\,.$$

Then we have: $Q = \bar{Q}$,

$$L_+ = \mathbb{Z}\alpha^1 + \mathbb{Z}\alpha^2 + \mathbb{Z}\alpha_3 + \mathbb{Z}\alpha_6\,, \qquad L_- = \mathbb{Z}\beta^1 + \mathbb{Z}\beta^2\,,$$

and

$$Q/L = \{L, \alpha_1 + L, \alpha_2 + L, \alpha_1 + \alpha_2 + L\}\,.$$

Hence, by Proposition 3.6,

$$\begin{aligned} V_Q^\sigma \cong & \big(V_{L_+} \otimes V_{L_-}^+\big) \oplus \big(V_{\frac{\alpha^1}{2}+L_+} \otimes V^+_{\frac{\beta^1}{2}+L_-}\big) \\ & \oplus \big(V_{\frac{\alpha^2}{2}+L_+} \otimes V^+_{\frac{\beta^2}{2}+L_-}\big) \oplus \big(V_{\frac{\alpha^1+\alpha^2}{2}+L_+} \otimes V^+_{\frac{\beta^1+\beta^2}{2}+L_-}\big)\,. \end{aligned}$$

There are 3 distinct irreducible V_Q^σ-modules of untwisted type:

$$\begin{aligned} & \big(V_{L_+} \otimes V_{L_-}^\pm\big) \oplus \big(V_{\frac{\alpha^1}{2}+L_+} \otimes V^\pm_{\frac{\beta^1}{2}+L_-}\big) \\ & \qquad \oplus \big(V_{\frac{\alpha^2}{2}+L_+} \otimes V^\pm_{\frac{\beta^2}{2}+L_-}\big) \oplus \big(V_{\frac{\alpha^1+\alpha^2}{2}+L_+} \otimes V^\pm_{\frac{\beta^1+\beta^2}{2}+L_-}\big)\,, \\ & \big(V_{L_+} \otimes V_{\mu_2+L_-}\big) \oplus \big(V_{\frac{\alpha^1}{2}+L_+} \otimes V_{\mu_2+\frac{\beta^1}{2}+L_-}\big) \oplus \big(V_{\frac{\alpha^2}{2}+L_+} \otimes V_{\mu_2+\frac{\beta^2}{2}+L_-}\big) \\ & \qquad \oplus \big(V_{\frac{\alpha^1+\alpha^2}{2}+L_+} \otimes V_{\mu_2+\frac{\beta^1+\beta^2}{2}+L_-}\big)\,, \end{aligned}$$

where $\mu_2 = \frac{1}{3}(\beta^1 + 2\beta^2)$. To describe the ones of twisted type, notice that the group G_σ associated to L_- is abelian and its characters χ are determined by $\chi(U_{\beta^i})$ where $i = 1, 2$. The latter are given by (2.19) with $s(\beta^i) = s_i \in \{\pm 1\}$. Thus we obtain 8 distinct irreducible V_Q^σ-modules of twisted type:

$$\begin{aligned} & \big(V_{L_+} \otimes V_{L_-}^{T_\chi,\pm}\big) \oplus \big(V_{\frac{\alpha^1}{2}+L_+} \otimes V_{L_-}^{T_\chi,\pm s_1}\big) \\ & \qquad\qquad \oplus \big(V_{\frac{\alpha^2}{2}+L_+} \otimes V_{L_-}^{T_\chi,\pm s_2}\big) \oplus \big(V_{\frac{\alpha^1+\alpha^2}{2}+L_+} \otimes V_{L_-}^{T_\chi,\pm s_1 s_2}\big)\,. \end{aligned}$$

We now describe the irreducible σ-twisted V_Q-modules using Section 2.5. The irreducible modules over the group G_σ associated to Q can be identified with the vector space $P = \mathbb{C}[q_1^{\pm1}, q_2^{\pm1}, q_3^{\pm1}, q_4^{\pm1}]$, so that:

$$\begin{aligned} U_{\alpha_1} &= q_1(-1)^{\frac{\partial}{\partial q_2}}\,, & U_{\alpha_2} &= q_2(-1)^{\frac{\partial}{\partial q_3}}\,, \\ U_{\alpha_3} &= q_3(-1)^{\frac{\partial}{\partial q_4}}\,, & U_{\alpha_6} &= q_4\,. \end{aligned}$$

In fact, there are 4 such modules depending on the signs of U_{β^i}; for each $s_1, s_2 \in \{\pm 1\}$, we have:

$$U_{\alpha_4} = -s_2 q_2 (-1)^{\frac{\partial}{\partial q_3}}\,, \qquad U_{\alpha_5} = -s_1 q_1 (-1)^{\frac{\partial}{\partial q_2}}\,,$$
$$U_{\alpha^i} = -s_i 4 q_i^2\,, \qquad U_{\beta^i} = s_i \frac{1}{16}\,.$$

The corresponding G_σ-module will be denoted P_χ where $\chi = (s_1, s_2)$. The automorphism σ acts on P_χ by:

$$\sigma(q_i) = -s_i q_i\,, \qquad i = 1, 2\,.$$

The irreducible σ-twisted V_Q-modules have the form $\mathcal{F}_\sigma \otimes P_\chi$, and they give rise to 8 irreducible V_Q^σ-modules of twisted type:

$$\big(\mathcal{F}_\sigma \otimes P_\chi\big)^{\pm}\,, \qquad \chi = (s_1, s_2)\,, \quad s_1, s_2 \in \{\pm 1\}\,.$$

They correspond to the above modules of twisted type with the same characters χ.

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Department of Mathematics, North Carolina State University, Raleigh, NC 27695, USA
E-mail address: bojko_bakalov@ncsu.edu

Department of Mathematics, Spring Hill College, Mobile, AL 36695, USA
E-mail address: jelsinger@shc.edu